The Xiao conjecture for trigonal fibrations

$f: S \to B$ smooth proj fibered surface.

$S_b \hookrightarrow S$ smooth fiber $\rightsquigarrow$ $\mathrm{Alb}(S_b) \to \mathrm{Alb}(S)$

Image is independent of b up to translation : A

$$0 \to A \to \mathrm{Alb}(S) \to \mathrm{Alb}(B) \to 0$$

All the S_b have A as a Jacobian isogeny factor.

- irregularity of S $\qquad q = h^0(S, \Omega^1) = \dim \mathrm{Alb}(S)$
- genus of S_b $\qquad g = h^0(S_b, K_{S_b}) = \dim \mathrm{Alb}(S_b)$
- relative irregularity of f $\qquad q_f = q - g(B) = \dim A$

Conjecture (Modified Xiao Conjecture Xiao '88, Pirola '92)

If $f: S \to B$ is a non-trivial fibered surface then

$$q_f \leq \frac{g}{2} + 1 \qquad (g \geq 2(q_f - 1))$$

(the original was $q_f \leq \frac{g+1}{2}$)

Ex: $\mathbb{Z}/3\mathbb{Z}$ Étale cover of a elliptic curve totally ramified in 3 pts.
Moduli space has dim 3.

$$\begin{array}{c} C \\ \downarrow {\scriptstyle \mathbb{Z}/3\mathbb{Z}} \\ E \end{array} \qquad E \to J(C) \to \underset{\dim 3}{A}$$

Known for:

- If $B \cong \mathbb{P}^1$ (Xiao 87)
- $\boxed{\text{Isotrivial fibrations}}$ (Serrano 96)
- Fibrations by hyp or bielliptic curves (Cai 98)
- Fibrations by curves of maximal Clifford index (BGAN 18) $\quad q_f \leq g - c_{\text{Cliff}}$
- " " plane quintics (FNP 18)

We will prove it for hyperelliptic fibrations.

Lemma: A hyperelliptic curve in an abelian variety is rigid up to translation.

(Lemma $\Longrightarrow$ Xiao)
for hyp. fibrations

$f : S \to B$ hyperelliptic fibration.

Let A be as above

We get a map $\varphi : S \longrightarrow A$ get family of hyperelliptic curves in A.

$\Longrightarrow$ either f is isotrivial or the family consists of covers of a fixed curve $C \subset A$.

non-degenerate

$$S_b \xrightarrow{d \geq 2} C \subset A$$
$$g(C) \geq \dim A$$

$$2g - 2 \geq 2(\overbrace{2g(C) - 2}) \geq 2(2\dim A - 2) = 4(\dim A - 1)$$

$$g \geq 2\dim A - 1 \Longrightarrow q_f = \dim A \leq \frac{g+1}{2}.$$

Proof of Lemma:

$C \subset A$ hyperelliptic, $\varphi : C \xrightarrow{2:1} \mathbb{P}^1$, $\varphi^{-1}(t) = \{x \neq x'\}$, t, $\{x, -x\}$ $\rightsquigarrow$

$$\begin{array}{ccccc} \mathbb{P}^1 & \to & \mathrm{Sym}^2 C & \to & \mathrm{Sym}^2 A \\ t & \mapsto & x + x' & & \downarrow + \\ & & & & A \end{array}$$

Maps $\mathbb{P}^1 \to A$ are constant so the composition is constant.

We will say C is "normalized" if the image is 0.

$$\left[C \subset A \text{ normalized hyp} \rightsquigarrow \mathbb{P}^1 \longrightarrow \left(\text{fiber of } \mathrm{Sym}^2 A \to A \text{ over } 0_A\right) = \{(a, -a) : a \in A\} = A/\pm \right]$$

So family of C's gives a covered surface $S \subset A/\pm$

Take $s = a/\pm \in S$ general.

$$\begin{array}{ccc} H^0(\widetilde{A/\pm}, \Omega^2) & \xrightarrow[\sim]{\text{eval}} & \Lambda^2 T^*_{\widetilde{A/\pm}, s} \\ \| & & \downarrow s \\ H^0(A, \Omega^2)^+ & & \\ \| & & \\ H^0(A, \Omega^2) & \xrightarrow[\sim]{\text{eval}} & \Lambda^2 T^*_{A,a} \end{array}$$

$\Rightarrow \exists\, \omega \in H^0(\widetilde{A/\pm}, \Omega^2)$ s.t. $\omega|_S \neq 0$.

$\square$

Thm (M)

Non-degenerate trigonal curves are rigid up to transl. in ab. varieties of dim ≥ 4.

$C_{\min}^{\text{trigonal}} \quad A^m \times B^n.$

Cor: The Xiao conjecture holds for trigonal fibrations.

Cor: " " " " " fibrations of genus ≤ 6.

Cor: A an abelian variety of dim ≥ 4.
$S \subset A$ a non-deg surface. $\varphi: S \dashrightarrow \mathbb{P}^2$ dominant.
Then $\deg\varphi \geq 4$. ($\min\{\deg\varphi : S \dashrightarrow \mathbb{P}^2$, dominant$\} = \operatorname{irr}(S)$)

Sketch:

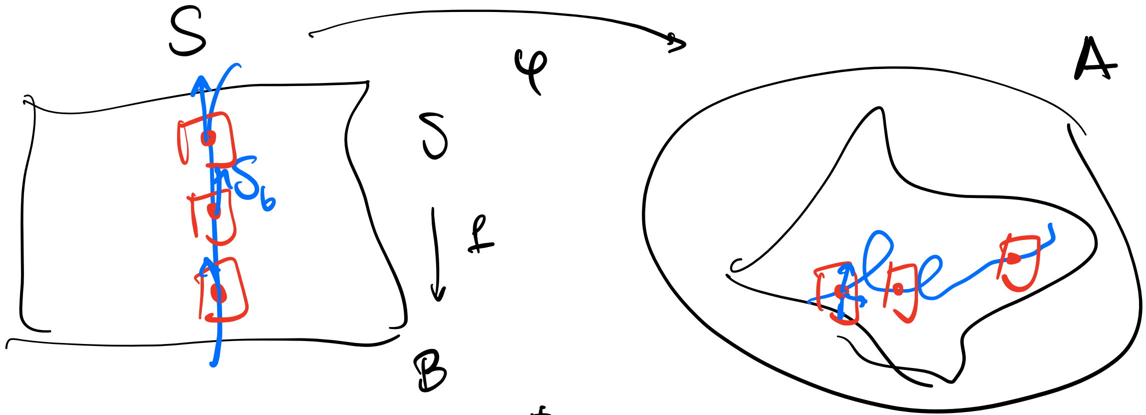
$C \subset A$ trigonal $\qquad \mathbb{P}^1 \longrightarrow \mathrm{Sym}^3 C \longrightarrow \mathrm{Sym}^3 A \longrightarrow A$

Normalize so the $\mathbb{P}^1$ lands in $\mathrm{Sym}^{3,0} A$, the fiber over 0_A.

Family of trigonal curves (normalized) $\rightsquigarrow$ Covered surface in A^3.

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & A \\ f \downarrow & & \\ B & & \end{array} \qquad \mathrm{Sym}^{3,0} A.$$

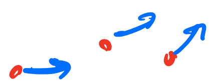
Have same hol 2-form.

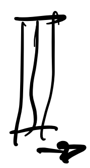


fiber of the map to $\mathbb{P}^1$

The vanishing of the 2-forms on $\mathrm{Sym}^{2,0} A$ on the unruled surface force these to be the same

$$\begin{array}{ccc} S_b & \longrightarrow & J(S_b) \\ & & \wr\downarrow \\ S_b & \longrightarrow & \mathbb{P}^{g-1} \end{array}$$





Contraction map.

$$\begin{array}{cccc} & & H^0(S_\Delta, \Omega^2_\Delta) & \\ & \nearrow & & \searrow \\ c: & \Lambda^2 H^0(A, \Omega^1) & \longrightarrow & H^0(S_b, K_{S_b}) \\ & \alpha \wedge \beta & \longmapsto & v \lrcorner (\alpha \wedge \beta) \end{array}$$